Exam Analysis on Manifolds

WIANVAR-07.2015-2016.2A

April 4th, 2016, 9:00-12:00 hrs.

This exam consists of **three** assignments. You get 10 points for free.

Assignment 1. (30 pt.)
Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$, and let $i : M \to \mathbb{R}^3$ be the inclusion map.

1. (8 pt.) Construct an atlas on M such that:

   (i) M becomes a two-dimensional C^∞-manifold;
   (ii) the inclusion map i is a C^∞-map.

   Prove that this atlas satisfies both (i) and (ii).

2. (8 pt.) Let σ be the one-form on $\mathbb{R}^3$ given by $\sigma = x\,dx + y\,dy$.
   Prove that $i^*\sigma = 0$.

3. (7 pt.) Let $\Omega = dx \wedge dy \wedge dz$ be the volume form on $\mathbb{R}^3$, and let X be the vector field on $\mathbb{R}^3$ given by

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

   Prove that $\iota_X\Omega = x dy \wedge dz - y dx \wedge dz$.
   (Recall that $\iota_X\Omega$ is the two-form given by $\iota_X\Omega(Y, Z) = \Omega(X, Y, Z)$.)

4. (7 pt.) Prove that $i^*(\iota_X\Omega)$ is a nowhere zero two-form on M.

Assignment 2. (30 pt.)
Let M be a compact connected $2n$-dimensional manifold without boundary, and let ω be a two-form on M.

In the following assignments, $\omega_k = \underbrace{\omega \wedge \cdots \wedge \omega}_{k \text{ factors}}$, for integers $k \geq 1$.

1. (8 pt.) Prove: if ω is closed, then for $k \geq 1$, the $2k$-form ω_k is also closed.

2. (8 pt.) Prove: if ω is exact, then for $k \geq 1$, the $2k$-form ω_k is also exact.

In the following two assignments, assume that ω_n is a nowhere zero $2n$-form on M.

3. (8 pt.) Prove that ω is not exact.

4. (6 pt.) Prove that M is not contractible to a point.

Assignment 3 on next page

Assignment 3. (30 pt.)
Let $M = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. We turn M into a Riemannian two-manifold by equipping it with the inner product $\langle \cdot, \cdot \rangle$, defined by

$$\langle v, w \rangle = \frac{v \cdot w}{h(x, y)^2},$$

for $v, w \in T_pM$, where $h : M \to \mathbb{R}$ is a positive C^2-function on M, and $v \cdot w$ is the standard inner product of v and w on $\mathbb{R}^2$.

1. (5 pt.) Prove that there is a differentiable function $f : M \to \mathbb{R}$ such that $\{F_1, F_2\}$, with $F_1 = f \frac{\partial}{\partial x}$ and $F_2 = f \frac{\partial}{\partial y}$, is an (orthonormal) moving frame on M.
2. (6 pt.) Determine the coframe $\{\vartheta_1, \vartheta_2\}$ of this moving frame (i.e., express these one-forms in terms of dx and dy).
3. (10 pt.) Determine the connection form ω_{12} of the moving frame $\{F_1, F_2\}$.
4. (9 pt.) Determine the Gaussian curvature $K(x, y)$ at $(x, y) \in M$, and prove that $K = -1$ for $h(x, y) = \frac{1}{2}(1 - x^2 - y^2)$.

Solutions

Assignment 1.

1. Let $U_1 = (0, 2\pi) \times \mathbb{R} \subset \mathbb{R}^2$ and let $U_2 = (-\frac{1}{2}\pi, \frac{1}{2}\pi) \times \mathbb{R} \subset \mathbb{R}^2$, and let $f_i : U_i \to \mathbb{R}^3$ be given by

$$f_i(u, v) = (\cos u, \sin u, v).$$

We shall prove that $\{(U_1, f_1), (U_2, f_2)\}$ defines a C^∞-atlas on M. First observe that $V := f_1(U_1) \cap f_2(U_2)$ consists of two connected components. More precisely, $(u, v) \in f_1^{-1}(V)$ iff $0 < u < \frac{1}{2}\pi$ or $\frac{3}{2}\pi < u < 2\pi$. Similarly, $(u, v) \in f_2^{-1}(V)$ iff $-\frac{1}{2}\pi < u < 0$ or $0 < u < \frac{1}{2}\pi$. Furthermore, if $f_1(u, v) = f_2(u', v')$, then $v' = v$ and $u' = u$ modulo 2π. A careful analysis shows that, for $(u, v) \in f_1^{-1}(V)$, we have

$$f_2^{-1} \circ f_1(u, v) = \begin{cases} (u, v), & \text{if } 0 < u < \frac{1}{2}\pi, \\ (u - 2\pi, v), & \text{if } \frac{3}{2}\pi < u < 2\pi. \end{cases}$$

Therefore, $f_2^{-1} \circ f_1 : f_1^{-1}(V) \to f_2^{-1}(V)$ is C^∞. Its inverse $f_1^{-1} \circ f_2 : f_2^{-1}(V) \to f_1^{-1}(V)$ is also C^∞.

To prove that the inclusion map is C^∞, observe that $\mathrm{id}^{-1} \circ i \circ f_j : U_j \to \mathbb{R}^3$ is equal to $g_j : U_j \to \mathbb{R}^3$, defined by $g_j(u, v) = (\cos u, \sin u, v)$. Note that on the manifold $\mathbb{R}^3$ we use the atlas $(\mathbb{R}^3, \mathrm{id})$. Also note that f_j and g_j only differ in the sense that they have different ranges. Since g_j is C^∞, the claim follows.

2. We shall prove that both local representatives $f_j^*(i^*\sigma)$, $j = 1, 2$, are zero. If f is either of the maps f_1 or f_2, then $f^*(i^*\sigma) = (i \circ f)^*(\sigma)$, and $(i \circ f)(u, v) = (\cos u, \sin u, v)$. Hence, $f^*(i^*\sigma) = \cos u \, d(\cos u) + \sin u \, d(\sin u) = -\cos u \sin u \, du + \sin u \cos u \, du = 0$.

3. Let $\omega = \iota_X \Omega$, then

$$\omega = \omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \, dx \wedge dy + \omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) \, dx \wedge dz + \omega(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \, dy \wedge dz.$$

Therefore, the claim follows from

$$\omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \Omega(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 0,$$

$$\omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) = \Omega(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z}) = -y,$$

$$\omega(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = \Omega(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = x.$$

4. The local representatives of ω are

$$(i \circ f_j)^*(\omega) = -\sin u \, d(\cos u) \wedge dv + \cos u \, d(\sin u) \wedge dv = du \wedge dv.$$

Since these local representatives are nowhere zero, ω is nowhere zero.

Assignment 2.

The proofs in parts 1 and 2 are by induction, using $\omega_k = \omega \wedge \omega_{k-1}$ for $k > 1$.

1. Since $d\omega = 0$, the claim is obvious for $k = 1$. So let $k > 1$, and suppose $d\omega_{k-1} = 0$. Then, since $\deg \omega_{k-1} = 2(k-1)$ is even:

$$d\omega_k = d\omega \wedge \omega_{k-1} + \omega \wedge d\omega_{k-1} = 0.$$

Therefore, ω_k is closed for all integers $k \geq 1$.

2. Since ω is exact (say $\omega = d\eta$), the claim is obvious for $k = 1$. So let $k > 1$, and suppose ω_{k-1} is exact. This implies $d\omega_{k-1} = 0$, so

$$\omega_k = d\eta \wedge \omega_{k-1} = d(\eta \wedge \omega_{k-1}) - \eta \wedge (d\omega_{k-1}) = d(\eta \wedge \omega_{k-1}).$$

Therefore, ω_k is exact for all integers $k \geq 1$.

3. Suppose ω is exact. According to part 2, ω_n is exact, say $\omega_n = d\eta$ for some $(2n-1)$-form η on M. The manifold M is orientable, since ω_n is a nowhere zero $2n$-form on M. Therefore, we may use the theorem of Stokes to derive

$$0 \neq \int_M \omega_n = \int_M d\eta = \int_{\partial M} \eta = 0,$$

since $\partial M = \emptyset$. (The first inequality follows from the fact that M is connected.) This contradiction proves that ω is not exact.

4. Since $d\omega_n = 0$ (being a $(2n+1)$-form on a $2n$-dimensional manifold), ω_n is closed. Suppose M is contractible to a point. Then ω_n is exact according to Poincaré's Lemma. According to Parts 2 and 3 this is a contradiction, so M is not contractible to a point.

Assignment 3.

1. Since $\langle f\frac{\partial}{\partial x}, f\frac{\partial}{\partial y}\rangle = 0$, and $\langle f\frac{\partial}{\partial x}, f\frac{\partial}{\partial x}\rangle = \langle f\frac{\partial}{\partial y}, f\frac{\partial}{\partial y}\rangle = \frac{f^2}{h^2}$, we take $f = h$.

2. Let ϑ_1 and ϑ_2 be the dual one-forms, then

$$\vartheta_1 = \vartheta_1(\frac{\partial}{\partial x})\, dx + \vartheta_1(\frac{\partial}{\partial y})\, dy = \frac{1}{h}\, dx.$$

Similarly: $\vartheta_2 = \frac{1}{h}\, dy$.

3. Let $\omega_{12} = P\, dx + Q\, dy$. To determine P and Q, we use the identities

$$d\vartheta_1 = \omega_{12} \wedge \vartheta_2 \quad \text{en} \quad d\vartheta_2 = -\omega_{12} \wedge \vartheta_1.$$

These yield

$$\frac{h_y}{h^2} dx \wedge dy = \frac{1}{h} P\, dx \wedge dy,$$

$$-\frac{h_x}{h^2} dx \wedge dy = \frac{1}{h} Q\, dx \wedge dy.$$

Therefore,

$$P = \frac{h_y}{h}, \quad Q = -\frac{h_x}{h}, \tag{1}$$

so

$$\omega_{12} = \frac{1}{h}\,(h_y\,dx - h_x\,dy).$$

4. Since $d\omega_{12} = -K\,\vartheta_1 \wedge \vartheta_2$, and

$$d\omega_{12} = (-P_y + Q_x)\,dx \wedge dy = h^2(-P_y + Q_x)\,\vartheta_1 \wedge \vartheta_2.$$

we have

$$K = h^2\,(P_y - Q_x). \tag{2}$$

Substitution in (1) and (2) yields

$$K = h\,(h_{xx} + h_{yy}) - (h_x^2 + h_y^2).$$

If $h(x, y) = \frac{1}{2}\,(1 - x^2 - y^2)$, then $K = -1$.